

## New evolution equations for turbulent boundary layers in arbitrary pressure gradients

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**Abstract.** A new approach to the classical closure problem for turbulent boundary layers is presented. This involves using the well-known mean-flow scaling laws such as Prandtl's law of the wall and the law of the wake of Coles together with the mean continuity and the mean momentum differential and integral equations. The important parameters governing the flow in the general non-equilibrium case are identified and are used for establishing a framework for closure. Initially, closure is done here empirically from the data but the framework is most suitable for applying the attached eddy hypothesis in future work. How this might be done is indicated here.

**Keywords.** Classical closure problem; turbulent boundary layers; attached eddy hypothesis; arbitrary pressure gradients.

### 1. Introduction

In earlier initial work, Perry *et al* (1994) derived a closure scheme for calculating the streamwise evolution of a turbulent boundary layer developing in a pressure gradient. The cases considered were not completely arbitrary and were confined to flows which could be considered to be in either approximate equilibrium or quasi-equilibrium. In approximate equilibrium flows, the Coles (1956) wake factor  $\Pi$  is assumed to be constant and the shear stress profiles are characterised only by  $\Pi$  (approximately), and in quasi-equilibrium flows,  $\Pi$  is allowed to vary but this variation is sufficiently slow so as to keep the parameter  $\zeta = S\delta_c(d\Pi/dx)$  small. Here  $\delta_c$  is the boundary layer thickness and  $S = U_1/U_\tau$  is a skin friction parameter where  $U_\tau$  is the local friction velocity and  $U_1$  is the local freestream velocity. This scheme was based in part on the law of the wall, the law of the wake and the mean momentum differential and integral equations together with continuity. The shear stress is found to be given by

$$\tau/\tau_0 = f_1[\eta, \Pi, S] + g_1[\eta, \Pi, S]\zeta + g_2[\eta, \Pi, S]\beta. \quad (1)$$

Here  $\tau$  is total shear stress,  $\tau_0$  is the wall shear stress,  $\eta = z/\delta_c$  where  $z$  is the wall-normal distance,  $\beta = (\delta^*/\tau_0)(dp/dx)$  is the Clauser (1954, 1956) pressure gradient parameter

where  $\delta^*$  is the displacement thickness,  $p$  is the freestream static pressure and  $x$  is stream-wise distance. Here  $f_1$ ,  $g_1$  and  $g_2$  are known universal analytical functions. Their precise form depends on the wall-wake formulation. Currently, the Lewkowicz (1982) formulation is used, i.e.

$$\frac{U}{U_\tau} = \frac{1}{\kappa} \log \left[ \frac{zU_\tau}{\nu} \right] + A + \frac{\Pi}{\kappa} W_c[\eta, \Pi], \quad (2)$$

where

$$W_c[\eta, \Pi] = 2\eta^2(3 - 2\eta) - (1/\Pi)\eta^2(1 - \eta)(1 - 2\eta).$$

The functions  $f_1$ ,  $g_1$  and  $g_2$  turn out to be rather long expressions covering several pages and were found using *Mathematica*. Details are given in Perry *et al* (1994) and Perry & Marusic (1995).

In the restricted flows considered by Perry *et al* (1994), the parameter  $\zeta$  is more or less neglected. Thus the problem reduces to considering the relation

$$C[\Pi, \beta, S] = 0, \quad (3)$$

where for a given  $\Pi$  it is assumed that the velocity defect distribution is fixed and the shear stress distribution is fixed. Hence, from data, if we know  $\beta$  at a given  $S$  for a fixed  $\Pi$  (i.e. for one experimental datum point), then for this fixed  $\Pi$  we can find  $\beta$  versus  $S$  for all  $S$  using (1) to ensure that  $\tau/\tau_0$  profiles are matched for all  $S$ . Now, it is found that for  $S$  sufficiently large,  $\beta = \beta_a$  (the asymptotic value of  $\beta$ ) and  $C$  is no longer a function of  $S$ . If this procedure is repeated for different values of  $\Pi$ , a one-to-one relationship between  $\beta_a$  and  $\Pi$  can be found which is based on experiment. This formulation is consistent with a universal relation for eddy viscosity  $\epsilon$ , i.e.  $\epsilon/(\delta_c U_\tau) = \phi[\eta, \Pi]$ . Unfortunately, such formulations are known to break down in non-equilibrium flows such as that considered by Marusic & Perry (1995).

## 2. New formulation

In order to account for non-equilibrium flows, the parameter  $\zeta$  must be added to our functional space and thus  $C$  in (3) needs to be replaced by,

$$\mathcal{F}[\Pi, S, \beta, \zeta] = 0. \quad (4)$$

It is assumed that no further parameters are involved in (4). Hence in order to describe the state of the layer, we require three of the four variables in the above expression. For quasi-equilibrium flows Perry *et al* (1994) relied on a one-parameter family to describe the shear stress, i.e.  $\tau/\tau_0 = f[\eta, \Pi]$  where  $f$  is assumed to be universal. A two-parameter family of shear stress profiles would be closer to the truth and when used in conjunction with (1) some information can be obtained regarding (4) as follows. Consider the  $S - \beta$  plane at a fixed  $\Pi$ . If such a plane contains an experimental datum point, then  $S$ ,  $\Pi$ ,  $\beta$  and  $\zeta$  are known for that datum point and so also is  $\tau/\tau_0$  versus  $\eta$  from (1). Trace out a curve for increasing  $S$  of fixed shear stress profile shape on the  $S - \beta$  plane. By taking  $S \rightarrow \infty$  we obtain asymptotic values of  $\zeta_a$  and  $\beta_a$  as shown in figure 1. Going to  $S = \infty$  is simply

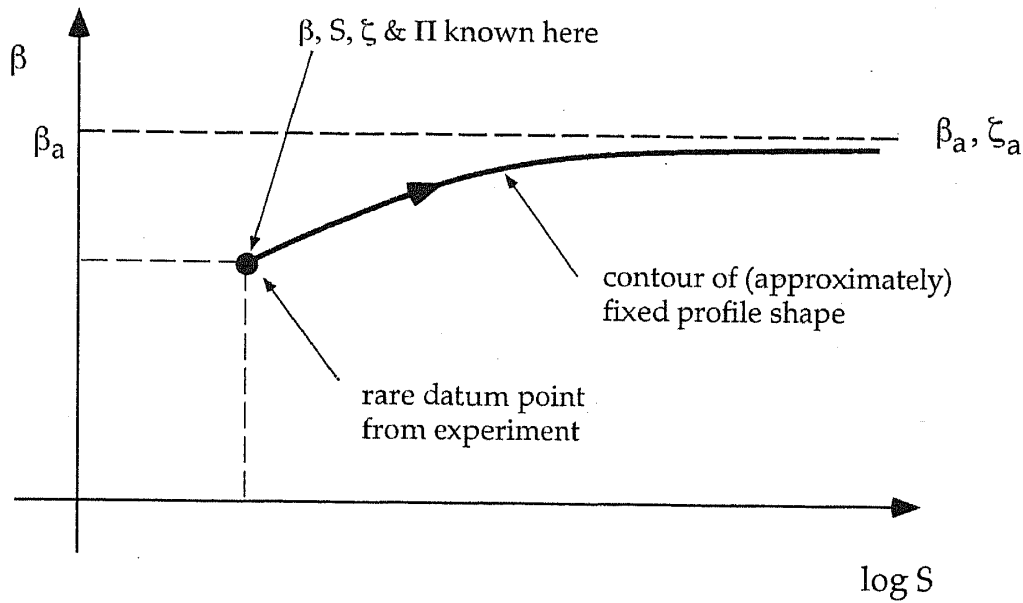


Figure 1. Contour of fixed shear stress profile shape for a fixed value of  $\Pi$ .

a convenient curve-fitting procedure and could never be approached experimentally. This process of keeping profile shape fixed will be referred to as “profile matching” and two methods have been tried and are described later. If this process is repeated often enough for different  $\Pi$ 's, then we obtain a  $\Pi - \beta_a$  diagram with distributions of extrapolated data points corresponding to different values of  $\zeta_a$ . By a surface fit to  $\zeta_a$  on the  $\Pi - \beta_a$  plane, contours of  $\zeta_a$  can be mapped out and we thus have a known function  $\psi$

$$\psi[\Pi, \beta_a, \zeta_a] = 0. \tag{5}$$

By shear stress profile matching we can then map out isosurfaces of  $\zeta$  in  $\Pi - \beta - S$  space. The first profile matching procedure tried here is an extension of the Perry *et al* (1994) method. The shear stress is broken into two parts and matched at  $\eta = m$  where  $m$  is a standard value of  $\eta$  which appears to give the best matching. From (1)

$$\left. \begin{aligned} & \frac{\tau}{\tau_0} = \left(\frac{\tau}{\tau_0}\right)_1 + \left(\frac{\tau}{\tau_0}\right)_2 \\ \text{with} & \left(\frac{\tau}{\tau_0}\right)_1 = f_1[m, \Pi, S] + g_2[m, \Pi, S]\beta \\ & = f_1[m, \Pi, \infty] + g_2[m, \Pi, \infty]\beta_a \\ \text{and} & \left(\frac{\tau}{\tau_0}\right)_2 = g_1[m, \Pi, S]\zeta \\ & = g_1[m, \Pi, \infty]\zeta_a. \end{aligned} \right\} \tag{6}$$

Choosing  $m = 0.4$  seems to give reasonable results. This method is crude but the results in most cases are not too different from a more sophisticated matching method treated later.

It is surprising how little reliable data are available. Most data fail the two-dimensional conservation of mean momentum test using (1). Using the data of Marusic & Perry (1995),

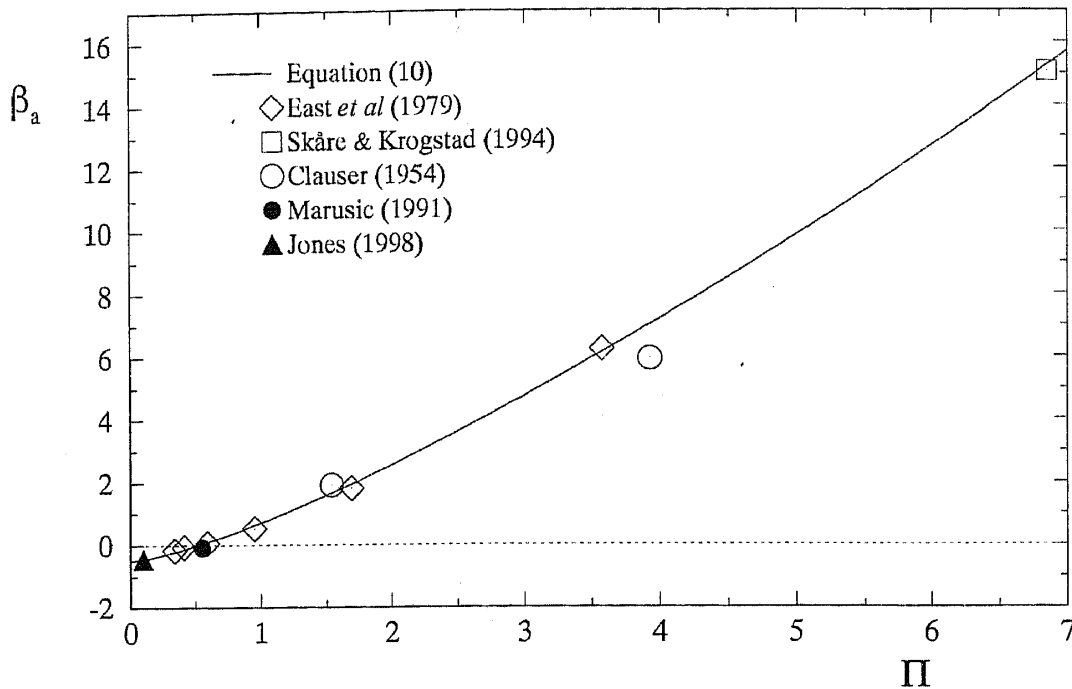


Figure 2. Comparison of (10) with experimental data which are in approximate equilibrium.

East *et al* (1979), Ludwig & Tillmann (1949) and Bradshaw & Ferriss (1965), a first tentative form for (5) has been obtained and is

$$\Delta\beta_a = 1.5 \exp(-1.15\Pi)\zeta_a + 0.051\zeta_a^2 + 0.0272\zeta_a^3, \quad (7)$$

for developing flows, i.e.  $\zeta_a > 0$ , and

$$\Delta\beta_a = \zeta_a \quad (8)$$

for relaxing flows;  $\zeta_a < 0$ . Here

$$\Delta\beta_a = \beta_a - \beta_{ae}, \quad (9)$$

where  $\beta_{ae}$  is the value of  $\beta_a$  for  $\zeta_a = 0$ .

It should be noted that there is a discontinuity of the  $\zeta_a$  surface slope on the  $\Delta\beta_a - \Pi$  plane at  $\Delta\beta_a = 0$ . This might reflect the possibility that the development and relaxation processes are quite different physically.

There have been various curve-fits for  $\beta_{ae}$  in the literature (although past work ignored the effect of the variable  $S$ ). We use here

$$\beta_{ae} = -0.5 + 1.21\Pi^{4/3} \quad (10)$$

which is close to an expression quoted by White (1974).

Equation (10) is shown in figure 2 compared to some equilibrium data and figure 3 shows various non-equilibrium data. Equations (7), (8), (9) and (10) are shown in figure 4.

### 3. Evolution equations

Here we extend the work of Perry *et al* (1994) to include the effect of the parameter  $\zeta$  in the evolution equations and we are now dealing with a two-parameter family of shear

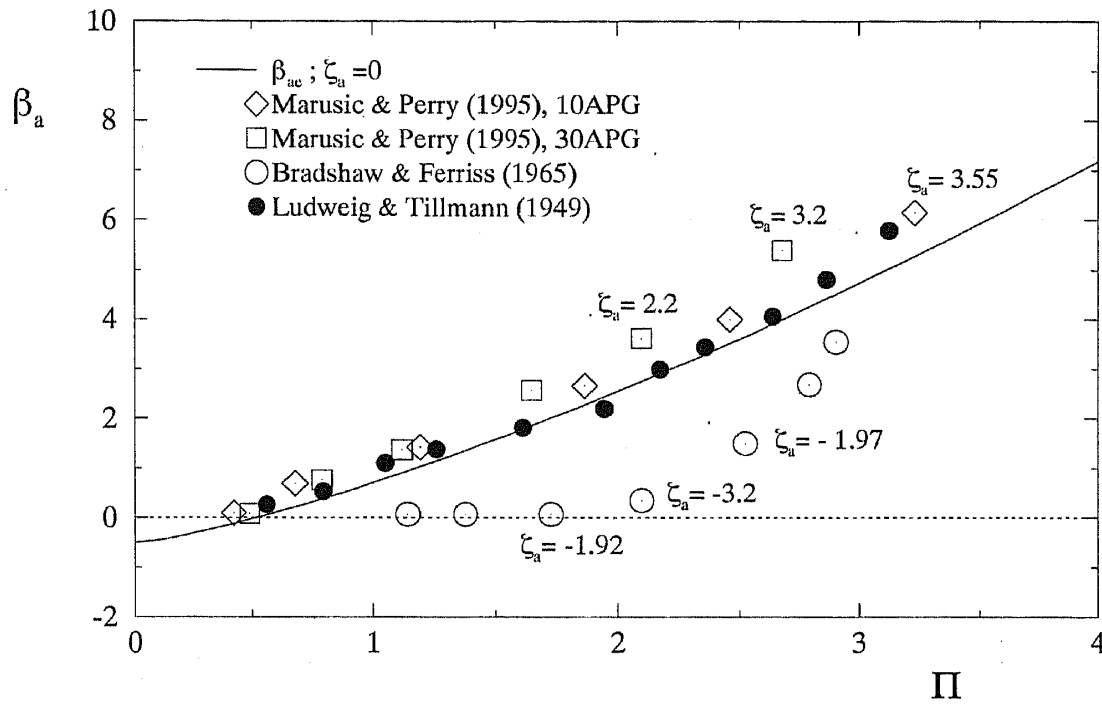


Figure 3. Comparison of (10) with non-equilibrium experimental data, i.e. with finite  $\zeta_a$  values. Some typical values of  $\zeta_a$  are shown for various data points.

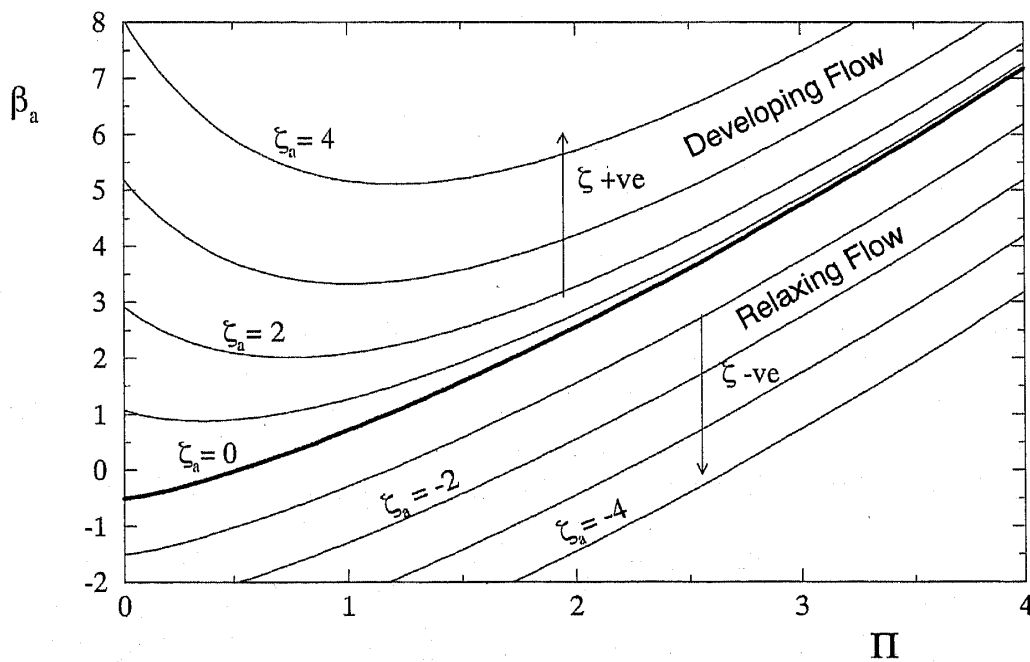


Figure 4.  $\beta_a$  versus Coles wake factor for different  $\zeta_a$  using (7), (8), (9), and (10).

stress profiles which can be expressed as

$$(\tau/\tau_0) = f[\eta, \Pi, \beta_a]. \quad (11)$$

After a considerable amount of algebra, the law of the wall, the law of the wake and the momentum integral equation give

$$\frac{dS}{dR_x} = \frac{\chi[R_x, K]R[S, \Pi, \zeta, \beta]}{SE[\Pi] \exp[\kappa S]}, \quad (12)$$

where

$$\begin{aligned} \kappa &= \text{Karman constant} = 0.41, \\ R &= \frac{S}{\kappa S^2 C_1 - \kappa S C_2 + C_2} + \frac{\beta(2SC_1 - C_2)}{C_1(\kappa S^2 C_1 - \kappa S C_2 + C_2)} \\ &\quad + \frac{\zeta((dC_2/d\Pi) - S(dC_1/d\Pi) - N(C_2 - SC_1))}{\kappa S C_1 - \kappa S C_2 + C_2}, \\ N &= W_c[1, \Pi] + \Pi \frac{dW_c[1, \Pi]}{d\Pi}, \end{aligned}$$

and for the Lewkowicz (1982) formulation  $N = 2$ ,

$$E[\Pi] = \exp \left[ -\kappa \left( A + \frac{\Pi}{\kappa} W_c[1, \Pi] \right) \right],$$

where  $A (= 5.0)$  is the logarithmic law of the wall constant,

$$\begin{aligned} C_1[\Pi] &= \int_0^\infty \frac{U_1 - U}{U_\tau} d\eta, \\ C_2[\Pi] &= \int_0^\infty \left( \frac{U_1 - U}{U_\tau} \right)^2 d\eta, \\ R_x &= \frac{xU_0}{\nu} \quad \text{and} \quad K = \frac{\nu}{LU_0}, \end{aligned}$$

where  $U_0$  is the value of the freestream velocity at some initial point  $R_x = 0$  or  $x = 0$ ,  $U_1$  is the freestream velocity at some general value of  $x$  or  $R_x$  and  $L$  is a characteristic length scale of the  $U_1$  distribution.  $K$  is often referred to as an acceleration parameter and  $\chi[R_x, K] = U_1/U_0$ ; usually  $\chi = \chi[R_x, K]$ .

From the law of the wall, law of the wake and the definition of  $\zeta$  we have

$$\frac{d\Pi}{dR_x} = \frac{\zeta \chi[R_x, K]}{S^2 E[\Pi] \exp[\kappa S]}. \quad (13)$$

From the law of the wall, law of the wake and the definition of  $\beta$  we have

$$S^2 E[\Pi] \exp[\kappa S] \frac{1}{\chi^2} \frac{d\chi}{dR_x} = -\frac{\beta}{C_1[\Pi]}. \quad (14)$$

Equations (12) and (13) form a set of coupled first order ODE's for  $S$  and  $\Pi$ . Equation (14) is an auxiliary equation where  $\chi$  is a known given function of  $R_x$  and  $K$ .

With the aid of (6), (7), (8), (9) and (10) a closure equation can be generated in the form (4), i.e.

$$\mathcal{F}[\Pi, S, \beta, \zeta] = 0.$$

The solution procedure is that at a given  $R_x$  and  $K$ ,  $\chi$  is known and, if  $\Pi$  and  $S$  are known, (14) will give  $\beta$ . Hence (4) will give  $\zeta$  and so (12) and (13) can be stepped by a  $\Delta R_x$  to give an updated  $\Pi$  and  $S$  and the process is thus repeated from the initial conditions  $S_0$  and  $\Pi_0$  at  $R_x = 0$ .

It turns out that for source, sink and zero pressure gradient flows,  $(d\chi/dR_x)/\chi^2$  in (14) is constant ( $= K$ ) and (12) and (13) effectively become autonomous, i.e.  $R_x$  need not appear explicitly with appropriate change of independent variable. This means that solutions can be displayed on a  $S - \Pi$  phase plane and solution trajectories can only cross at critical points as in standard phase-plane portrait analysis. Some examples of this are shown in figures 5, 6 and 7.

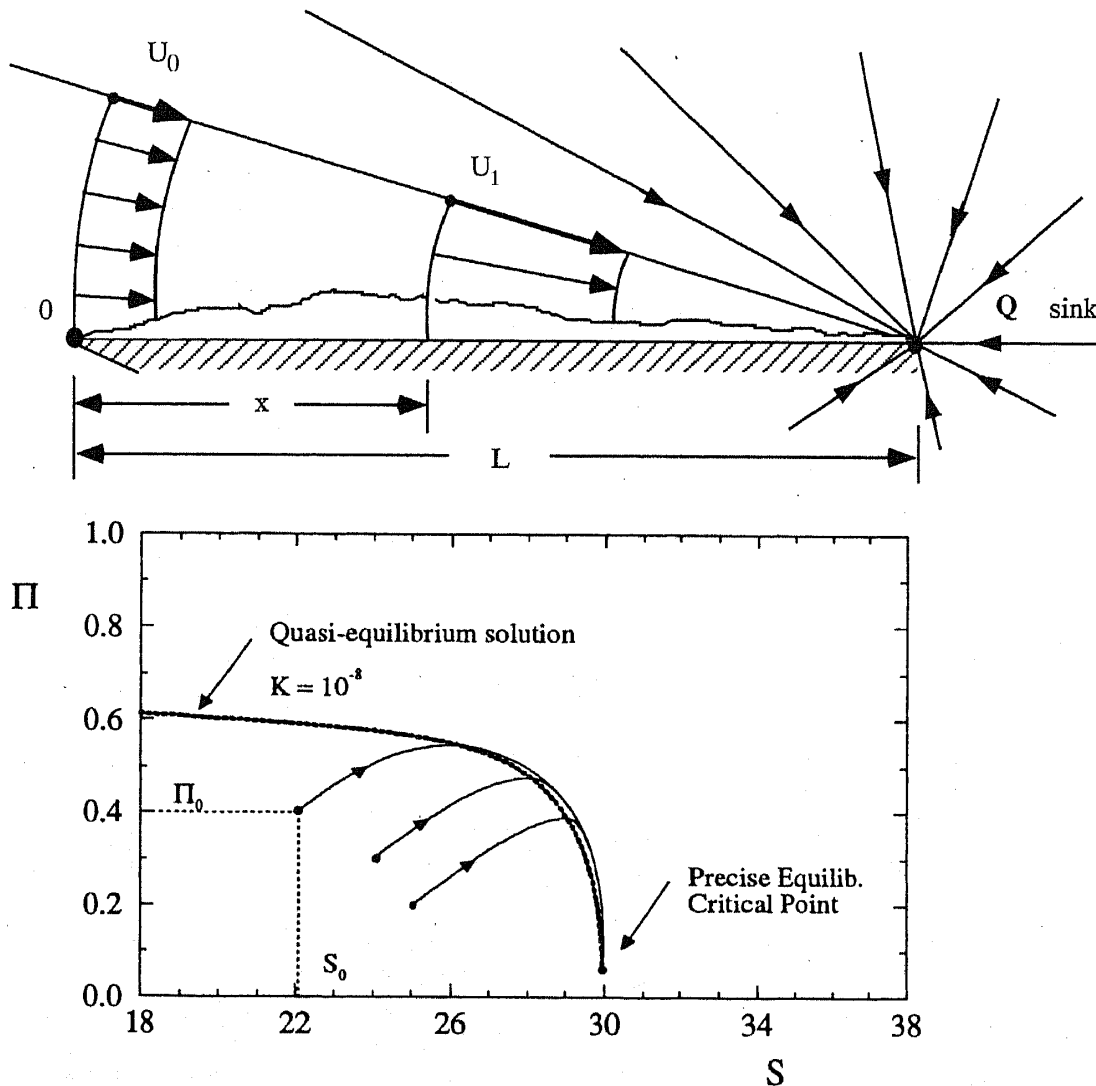


Figure 5. Sink flow.

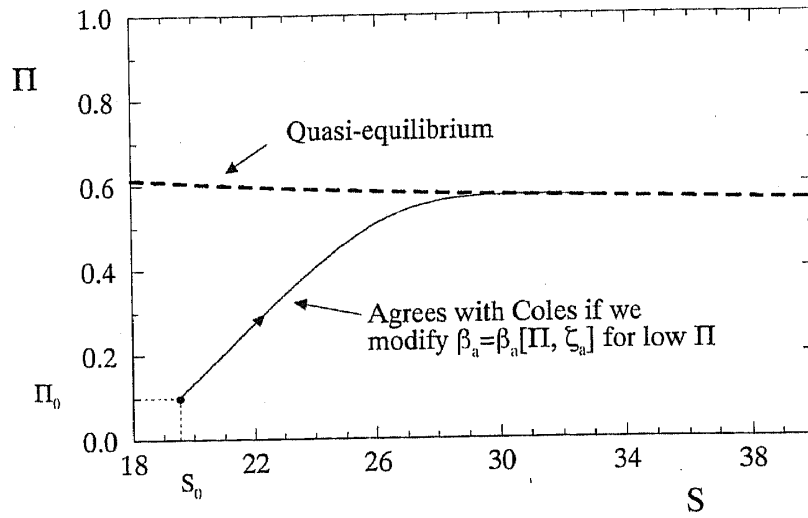


Figure 6. Zero pressure gradient flow.

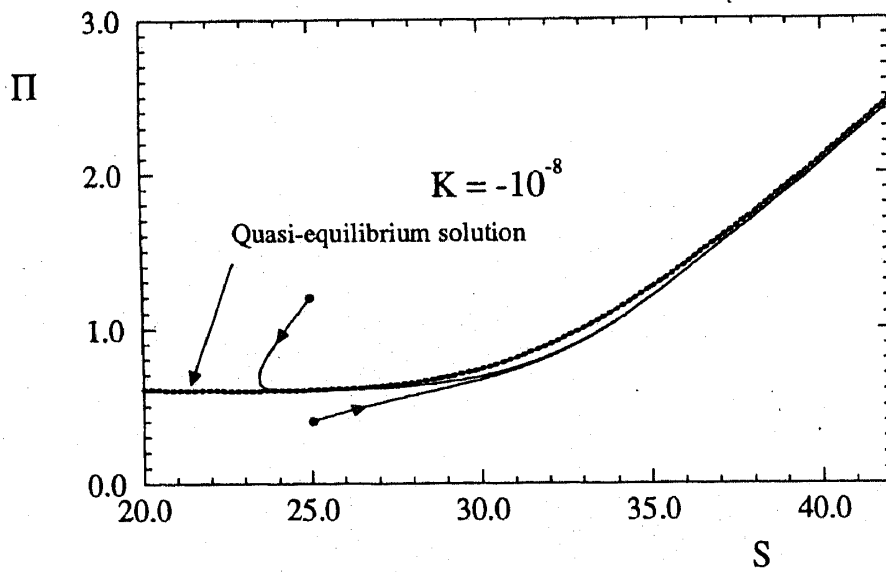
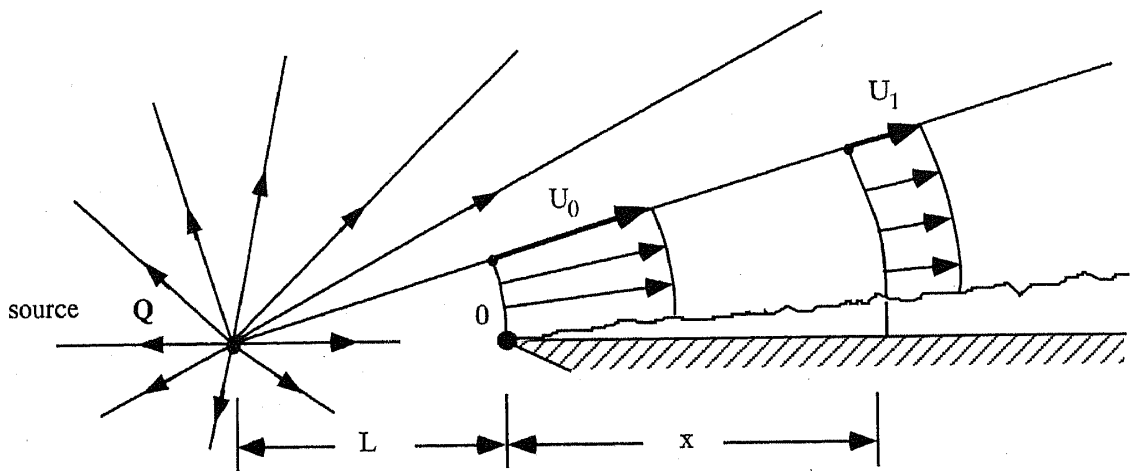


Figure 7. Source flow.

It is interesting to note that for sink flow, precise equilibrium is achieved at a critical point where  $d\Pi/dR_x$  and  $dS/dR_x$  are simultaneously zero as predicted by Rotta (1962) and Townsend (1976). The quasi-equilibrium solutions of Perry *et al* (1994) are shown for comparison. Note that for the solutions of the low  $\Pi$  for the zero pressure gradient flows to agree with the data cited by Coles (1962), an extra term,  $8 \exp(-3\Pi^2)\zeta_a$ , has been added to the RHS of (7) and this has influence at  $\Pi < 1.5$ . This goes to show how tentative all of these formulations are, being based on such limited data. Nevertheless, this exercise helps develop the mathematical machinery.

#### 4. A more sophisticated matching

Matching shear stress profiles with a least squares error criterion gives more accurate fits. Some trials have been carried out for the Marusic & Perry (1995) data. Here  $\beta_a$  and  $\zeta_a$  are found by optimising for least square error, i.e. given  $\tau/\tau_0$  at some data point,  $(\tau/\tau_0)_{S \rightarrow \infty}$  is found and

$$\frac{\partial}{\partial \beta_a} \left\{ \int_0^1 \left[ \left( \frac{\tau}{\tau_0} \right)_{S \rightarrow \infty} - \left( \frac{\tau}{\tau_0} \right) \right]^2 d\eta \right\} = 0,$$

$$\frac{\partial}{\partial \zeta_a} \left\{ \int_0^1 \left[ \left( \frac{\tau}{\tau_0} \right)_{S \rightarrow \infty} - \left( \frac{\tau}{\tau_0} \right) \right]^2 d\eta \right\} = 0. \tag{15}$$

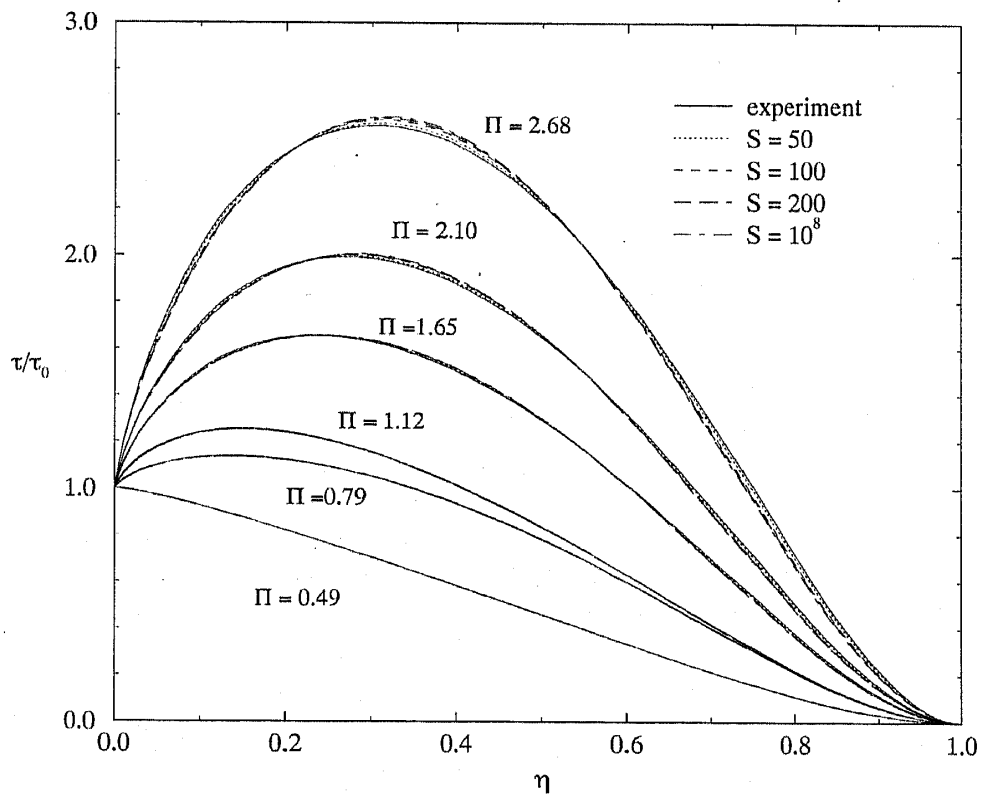
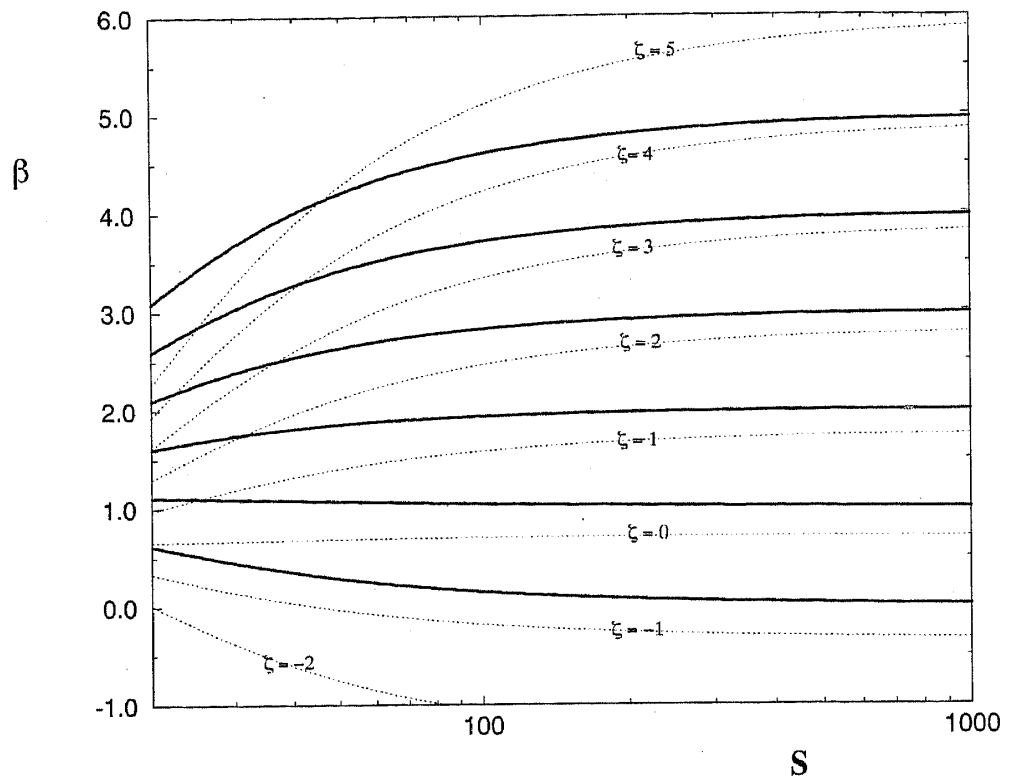
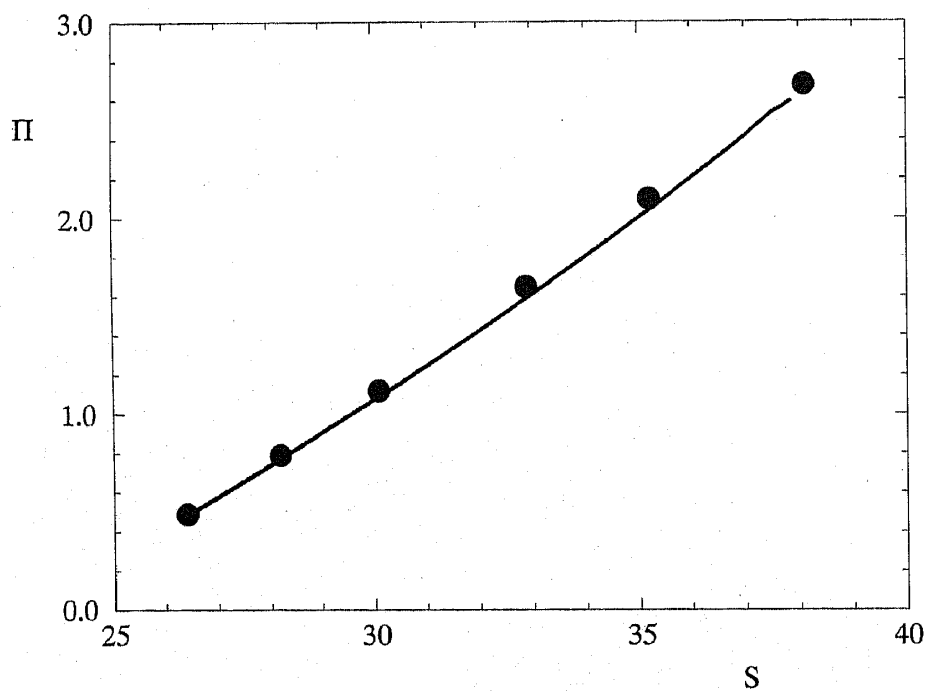


Figure 8. Matched shear stress profiles using least squares error for data of Marusic & Perry (1995) for different values of  $S$ .



**Figure 9.** Heavy lines indicate contours of fixed profile shape using least squares error matching. Dotted lines show contours of  $\zeta$ .  $\Pi = 1.0$  in all cases.



**Figure 10.** Data of Marusic & Perry (1995), 30APG flow. Solid line is computed solution of (12) and (13) using (14), (16) and (17).

This gives

$$\left. \begin{aligned} A_1[\Pi, S] + B_1[\Pi, S]\zeta + C_1[\Pi, S]\beta &= D_1[\Pi]\zeta_a + E_1[\Pi]\beta_a \\ A_2[\Pi, S] + B_2[\Pi, S]\zeta + C_2[\Pi, S]\beta &= D_2[\Pi]\zeta_a + E_2[\Pi]\beta_a \end{aligned} \right\}, \quad (16)$$

where  $A_1, A_2$  etc are all known analytical functions.

Figure 8 shows how well the profiles match for fixed  $\Pi$  over  $S$  from experimental values to  $S = 10^8$  for the data of Marusic & Perry (1995) case APG30. Figure 9 shows typical contours of fixed profile shape and contours of  $\zeta$  for a fixed  $\Pi = 1.0$ . By extrapolating to  $S \rightarrow \infty$ , a very limited and local surface fit for this data gives

$$\zeta_a = (0.727 + 0.231\Pi)\Delta\beta_a. \quad (17)$$

Using (16) and (17) for generating the function for (4), (12) and (13) were solved in a similar manner as earlier. This represents a non-autonomous case and the original data are compared with computed results as shown in figure 10. Hence we have shown at least that all the mathematical machinery is working.

### 5. Possible application of the attached eddy hypothesis

It is quite apparent that the amount of data available is quite limited and is extremely sparse. Furthermore, it has been found that for  $\Pi > 3$ , the least squares error profile matching procedure becomes ill-conditioned and either more data will be required or a model is needed. At low  $\Pi$ , i.e.  $\Pi < 0.5$ , there are very few data available. The profile-matching technique is merely a means for interpolation and extrapolation based on the assumption of a two-parameter family for the shear stress profiles and that  $\mathcal{F}[\Pi, S, \beta, \zeta] = 0$ . A realistic model would go a long way towards improving this interpolation and extrapolation even if the theory were incomplete and required empirical universal constants. Perry *et al* (1994) attempted to apply the attached eddy hypothesis for determining  $\beta_{ae} = \beta_{ae}[\Pi]$  with some limited success. The solution was numerical rather than analytical and was based on what Perry & Marusic (1995) referred to "wall-attached eddies". Since then it has become clear from this above work that two types of eddy structures are needed – namely wall eddies whose vortex cores reach to the wall and wake eddies whose vortex cores undulate in the spanwise direction but do not extend to the wall. The wall eddies are referred to as type A and the wake eddies are referred to as type B. Type B eddies are responsible for the mean wake component of the velocity defect law and for the wake contribution to the Reynolds shear stress. The wall eddy contribution to the Reynolds shear stress is that given by what Coles (1957) calls pure wall flow, which is equivalent to (according to the authors' interpretation)  $\beta = \beta_{ae} = -0.5$ ;  $\Pi = 0$  &  $S \rightarrow \infty$  and this is substituted into (1) to obtain the wall eddy contribution of shear stress.

What follows is an extension of Perry & Marusic (1995). The velocity defect is

$$U_D^* = \frac{U_1 - U}{U_\tau} = f[\eta, \Pi], \quad (18)$$

and for the wake component the model gives a convolution integral

$$\frac{dU_{DB}^*}{d\lambda_E} = \int_{-\infty}^{\infty} h_B[\lambda]e^{-\lambda}T_B[\lambda - \lambda_E]w_B[\lambda - \lambda_E]d\lambda, \quad (19)$$

where  $\lambda_E = \ln(\delta_c/z)$  and  $\lambda = \ln(\delta/z)$  where  $\delta$  is the eddy length scale.  $T_B$  is a weighting factor dependent on the characteristic velocity scale of an eddy of length scale  $\delta$  and  $w_B$  is a weighting factor dependent on the wall surface area population density of eddies of length scale  $\delta$ . Here  $h_B[\lambda]$  is a function of representative eddy shape which we assume is fixed. Let us put

$$\left. \begin{aligned} T_B[\lambda - \lambda_E] &= T_c \psi[\xi] \\ \text{and } w_B[\lambda - \lambda_E] &= w_c \phi[\xi] \end{aligned} \right\} \quad (20)$$

where  $T_c$  and  $w_c$  are weighting factor coefficients,  $\xi = \lambda - \lambda_E = \ln(\delta/\delta_c)$  and  $\psi$  and  $\phi$  are assumed to be functions of  $\xi$  alone. A structural similarity is implied here which would need checking in later work. Integrate (19) to give

$$U_{DB}^*[0] = \frac{2\Pi}{\kappa} = T_c w_c \int_0^1 \frac{1}{\eta} \int_{-\infty}^{\infty} h_B[\lambda] e^{-\lambda} \psi[\xi] \phi[\xi] d\lambda d\eta. \quad (21)$$

The integral is a universal number  $\mathcal{N}$  (say) and so

$$T_c w_c = (2\Pi/\kappa\mathcal{N}). \quad (22)$$

Now the Reynolds shear stress wake component is given by

$$-\left(\frac{\overline{u_1 u_3}}{U_\tau^2}\right)_B = \int_{-\infty}^{\infty} I_{13B}[\lambda] T_B^2[\lambda - \lambda_E] w_B[\lambda - \lambda_E] d\lambda, \quad (23)$$

where  $I_{13B}$  is the Townsend (1976) eddy intensity function and depends on representative eddy shape. Integrate to give

$$\int_0^1 -\left(\frac{\overline{u_1 u_3}}{U_\tau^2}\right)_B d\eta = T_c^2 w_c \int_0^1 \int_{-\infty}^{\infty} I_{13B}[\lambda] \psi^2[\xi] \phi[\xi] d\lambda d\eta, \quad (24)$$

and so

$$\int_0^1 -\left(\frac{\overline{u_1 u_3}}{U_\tau^2}\right)_B d\eta = M T_c^2 w_c, \quad (25)$$

where  $M$  is a universal number as given by the double integral on the RHS of (24).

Now it can be shown that from (1) that for  $S \rightarrow \infty$

$$\int_0^1 \left(\frac{\tau}{\tau_0}\right)_B d\eta \approx K(\beta_{ae} + 1/2), \quad (26)$$

i.e. it has only a very weak dependence on  $\Pi$  and  $K$  is a universal constant.

Hence from (25) and (26) we have

$$K(\beta_{ae} + 1/2) = M T_c^2 w_c. \quad (27)$$

We now postulate (purely because it seems plausible and appears to agree with experiment) that

$$T_c^2/w_c = P, \quad (28)$$

where  $P$  is a universal constant for all equilibrium layers, i.e. for all  $\beta_{ae}$ . If we assume power law relations, i.e.

$$\left. \begin{aligned} T_c &\propto \Pi^a \\ w_c &\propto \Pi^b \end{aligned} \right\}, \quad (29)$$

then from (22) and (28) we obtain  $a = 1/3$  and  $b = 2/3$ , i.e.

$$T_c^2 w_c \propto \Pi^{4/3}.$$

Hence from (27)

$$\beta_{ae} + 1/2 = Q\Pi^{4/3}, \quad (30)$$

where  $Q$  is a universal constant. From experiment  $Q = 1.21$  (using the profile matching given by (6)) and so

$$\beta_{ae} = -1/2 + 1.21\Pi^{4/3} \quad (31)$$

or

$$\Pi = 0.86(\beta_{ae} + 1/2)^{0.75}, \quad (32)$$

compared with White (1974) where

$$\Pi = 0.8(\beta_{ae} + 1/2)^{0.75}. \quad (33)$$

Now a more complete solution could be found from (1) as follows

$$\int_0^1 \left( \frac{\tau}{\tau_0} \right)_B d\eta = f_{1i}[\Pi, \infty] - f_{1i}[0, \infty] + g_{2i}[\Pi, \infty]\beta_{ae} + g_{2i}[0, \infty]\frac{1}{2}. \quad (34)$$

Here the suffix  $i$  means integrated with respect to  $\eta$  from 0 to 1, and the above is equal to  $\mathcal{L}\Pi^{4/3}$  from the attached eddy hypothesis where  $\mathcal{L}$  is a universal constant. Therefore

$$\beta_{ae} = \{\mathcal{L}\Pi^{4/3} - f_{1i}[\Pi, \infty] + f_{1i}[0, \infty] - g_{2i}[0, \infty]\frac{1}{2}\}/g_{2i}[\Pi, \infty]. \quad (35)$$

If we make  $\mathcal{L} = 0.71$  this is indistinguishable from (31).

As a first attempt to include  $\zeta_a$  effects, the appropriate term in (1) could be included so that (35) reads

$$\beta_a = \{\mathcal{L}\Pi^{4/3} - f_{1i}[\Pi, \infty] + f_{1i}[0, \infty] - g_{1i}[\Pi, \infty]\zeta_a - g_{2i}[0, \infty]\frac{1}{2}\}/g_{2i}[\Pi, \infty]. \quad (36)$$

Although (36) gives a family of curves for different  $\zeta_a$  on the  $\Pi - \beta_a$  plane, it is essentially equivalent to a universal distribution of eddy viscosity because of the assumptions behind (25). This implies that  $\mathcal{L}$  is a constant. In reality  $\mathcal{L}$  should be a function of  $\zeta_a$  and  $\Pi$ . More physics will be needed to derive this function.

## 6. Conclusions and discussion

A framework has been constructed for formulating closure for a turbulent boundary layer evolving in an arbitrary streamwise pressure gradient. The framework shows promise for applying the attached eddy hypothesis although this hypothesis would need further development to include the effects of the parameter  $\zeta_a$ . Further experiments used in conjunction with the above framework would be helpful. The above study is still in an early stage of development but much of the mathematical machinery is working.

Evolution equations based on an integral approach are consistent with the attached eddy hypothesis which utilizes convolution integrals. The above approach seems more reasonable physically than most differential field methods because, in reality, the transport properties at one point in the flow must be intimately related to motions remote from that point. This feature is an important aspect of the attached eddy hypothesis.

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